Lattice gauge notes VI

- two flavors: three parameters, m_u, m_d and θ
- correspond to: tilt, warp, and angle between them

- back to the lattice, K near critical should mimic
- parity broken for odd flavors, $K > K_c$
- 2 flavors and the Aoki phase – follow chiral review

The Ginsparg Wilson relation

I begin by considering the Fermionic part of some action as a quadratic form

$$S_f = \sum_i \overline{\psi} D \psi.$$

The usual "continuum" Dirac operator $D = \sum \gamma_\mu D_\mu$ naively anti-commutes with γ_5, i.e. $[\gamma_5, D]_+ = 0$. Then the change of variables $\psi \to e^{i\theta\gamma_5}\psi$ and $\overline{\psi} \to \overline{\psi} e^{i\theta\gamma_5}$ would be a symmetry of the action. This, however, is inconsistent with the chiral anomalies. The conventional continuum discussions map this phenomenon into the Fermionic measure (Fujikawa, 1979).

On the lattice we work with a finite number of degrees of freedom; thus, the above variable change is automatically a symmetry of the measure. To parallel the continuum discussion, it is necessary to modify the symmetry transformation so that the measure is no longer invariant. Remarkably, it is possible to construct actions exactly invariant under the altered symmetries.

To be specific, one particular modification (Neuberger, 1998b,c; Luscher, 1998; Chiu and Zenkin, 1999; Chandrasekharan, 1999) that leads to interesting consequences starts with the change of variables

$$\begin{aligned} \psi &\longrightarrow e^{i\theta\gamma_5(1+aD)}\psi \\ \overline{\psi} &\longrightarrow \overline{\psi} e^{i\theta\gamma_5} \end{aligned}$$

where a represents the lattice spacing. Note the asymmetric way in which the independent Grassmann variables ψ and $\overline{\psi}$ are treated. Requiring the action to be unchanged gives the relation (Ginsparg and Wilson, 1982; Hasenfratz, Laliena, and Niedermayer, 1998; Hasenfratz, 1998)

$$\gamma_5 D + D\gamma_5 + aD\gamma_5 D = 0.$$

I also assume the Hermeticity condition $\gamma_5 D \gamma_5 = D^\dagger$. The "Ginsparg-Wilson relation" along with the Hermititicity condition is equivalent to the unitarity of the combination $V = 1 + aD$.

Neuberger (1998b,c) and Chiu and Zenkin (1999) suggested a simple construction of an operator satisfying this condition. For this an appropriate operator V could be found via a unitarization of an undoubled but chiral symmetry violating Dirac operator, such as the Wilson operator D_w. This operator should also satisfy the above Hermeticity condition. From this build

$$V = -D_w(D_w^\dagger D_w)^{-1/2}.$$

More precisely, find a unitary operator to diagonalize $D_w^\dagger D_w$, take the square root of the eigenvalues, and undo this unitary transformation.

At this point the hopping parameter in D_w is a parameter. To have the desired single light Fermion per flavor of the theory, the hopping parameter should be appropriately adjusted to lie above the critical value where D_w describes a massless flavor, but not so large that additional doublers come into play (Neuberger, 1999; Golterman and Shamir, 2000). There are actually two parameters to play with, the hopping parameter of D_w, and the lattice spacing. When the latter is finite and gauge fields are present, the location of the critical hopping parameter in D_w is expected to shift from that of the free Fermion theory. There is potentially a rather complex phase structure in the plane of these two parameters, with various numbers of doublers becoming exactly massless modes. The Ginsparg-Wilson relation in and of itself does not in general determine the number of massless Fermions.

Although the Wilson operator entering this construction is local and quite sparse, the resulting action is not; it involves direct couplings between arbitrarily separated sites (Hernandez, Jansen and Luscher, 1999; Horvath, 1998, 1999). How rapidly these couplings fall with distance depends on the gauge fields and is not fully understood. The five dimensional domain-wall theory is local in the most naive sense of the word; all terms in the action only couple nearest neighbor sites. Were one to integrate out the heavy modes, however, the resulting low energy effective theory would also involve couplings with arbitrary range. Despite these non-localities, recent encouraging studies (Neuberger 1998c; Edwards, Heller, and Narayanan, 1999; Borici, 1999; Hernandez, Jansen and Lellouch, 2000; Dong, Lee, Liu, and Zhang, 2000; Gattringer, 2000) show that it may indeed be practical to implement the required inversion in large scale numerical simulations. The overlap operator should have memory advantages since a large number of fields corresponding to the extra dimension do not need to be stored. The extent to which this outweighs the additional complexity in implementation remains to be determined.

This approach hides the infinite sea of heavy Fermion states alluded to above. It is implicit in the presence of zero modes in the inversion. This is directly related to the connection with the index theorems in the continuum; for recent reviews see Adams (2000) and Kerler (2000). Recent detailed analysis (Luscher, 2000; Kikukawa and Yamada, 1999) shows that this operator is particularly well behaved order by order in perturbation theory. This has led to hopes that this may lead the way to a rigorous formulation of chiral models, such as the standard model.

When the mass is turned on, the chiral symmetry with an even number of flavors should make the sign of the mass irrelevant. A combination that becomes its negative under the above transformations with an angle of pi is

$$m\overline{\psi}(1-D/2)\psi$$

Notes on an overlap Hamiltonian

In these notes I explore a possible Hamiltonian version of the overlap operator. I start directly with continuum time; this is not an attempt to find a transfer matrix from a discrete time overlap formula. To do the latter sounds hard since the overlap operator is not local. To do the reverse, i.e. to go from continuous to discrete time, should proceed just as in my old papers involving a Wilson projection in the time-like direction. The resulting operator will be local in time, but not in space.

To set things up let me review the continuum Hamiltonian, which is

$$H_c = \int dx\ \psi^\dagger \gamma_0(\vec{\gamma}\cdot\vec{D}+m)\psi$$

with the canonical commutation relations

$$[\psi(x),\psi^\dagger(y)]_+ = \delta(x,y)$$

and I use Hermitian gamma matrices. In momentum space

$$H_c = \int \frac{dp}{2\pi}\ \psi^\dagger \gamma_0(i\vec{\gamma}\cdot\vec{p}+m)\psi$$

Charge conservation is manifested in the fact that H_c commutes with

$$U = e^{i\theta\int dx\ \psi^\dagger\psi} = e^{i\theta Q}$$

For axial symmetry we want to use

$$U_A = e^{i\theta\int dx\ \psi^\dagger\gamma_5\psi} = e^{i\theta Q_5}.$$

The mass term is not invariant under this, and of course with gauge fields present the anomaly comes in.

Define D_c by

$$H_c = \psi^\dagger\gamma_0 D_c\psi$$

with the integral over space implicit. The space for D_c is square integrable spinor functions. This operator satisfies

$$\gamma_5 D_c\gamma_5 = D_c^\dagger$$
$$\gamma_0 D_c\gamma_0 = D_c^\dagger$$

both properties that will carry over to the lattice. However, the continuum property $[D_c, D_c^\dagger] = 0$ will not carry over to the Wilson Hamiltonian. For that case D_c can have different left and right eigenvalues and the eigenvectors are not in general orthogonal. On the other hand, the overlap operator constructed below is diagonalizable since it is constructed from a unitary operator.

The free continuum eigenvalues of D_c are $\lambda = m \pm i|p|$ with an infinite range for the momentum. For every imaginary eigenvalue there is a complex conjugate one obtained by applying either γ_0 or γ_5 to the eigenvector:

$$\begin{aligned} D_c\phi &= \lambda\phi \\ D_c\gamma_5\phi &= \gamma_5 D_c^\dagger\phi = \lambda^*\gamma_5\phi \end{aligned}$$

This is still true with the gauge fields turned on.

The eigenvectors for D_c are different than those for the Hermitean operator $h \equiv \gamma_0 D_c$ since these objects do not commute. The latter can be found via $(\gamma_0 D_c)^2 = D_c^\dagger D_c$, giving free energy eigenvalues $\pm\sqrt{p^2+m^2}$.

Since $\gamma_0\gamma_5 h = -h\gamma_0\gamma_5$, if $h\phi = E\phi$, then $h\gamma_0\gamma_5\phi = -E\gamma_0\gamma_5\phi$ and energy eigenvalues always occur in positive/negative energy pairs. This is still true when gauge fields are present.

There seems to be one more operator to study, $h_5 \equiv \gamma_5 D_c$, which is Hermitean and has the same eigenvalues as $h_0 \equiv \gamma_0 D_c$ but different eigenvectors, again since they don't commute. The operators h and h_5 anticommute.

Now for the lattice, start with the Wilson Hamiltonian, written in the form

$$H_W = \psi^\dagger\gamma_0 D_W\psi.$$

The sum over space is implicit here. I still have the analog properties from the continuum:

$$\begin{aligned} \gamma_5 D_W\gamma_5 &= D_W^\dagger \\ \gamma_0 D_W\gamma_0 &= D_W^\dagger \\ [D_W, D_W^\dagger] &= 0 \end{aligned}$$

Note the symmetry between γ_0 and γ_5.

For free fields in momentum space, I write

$$D_W = M + \sum_j (i\gamma_j \sin(q_j) + (1-\cos(q_j)))$$

The critical case is $M=0$. Without gauge fields this can be diagonalized to give eigenvalues

$$\lambda_\pm(q_j) = \pm i\sqrt{\sum_j \sin^2(q_j)} + M + \sum_j (1-\cos(q_j))$$

This is kind of like a superposition of a bunch of circles. The real eigenvalues are in the set $\{M, M+2, M+4, M+6\}$. As for the continuum case, γ_5 relates eigenvalues with their complex conjugates.

Now try to do something like Herbert does with GW, defining

$$H = \psi^\dagger \gamma_0 D \psi$$

and construct

$$\begin{aligned} V &= D_W (D_W^\dagger D_W)^{-1/2} \\ D &= V+1 \end{aligned}$$

Depending on the choice of M, this can have robust zero energy states. Since V is unitary, these occur for eigenvalues of V near -1. It would appear that near the continuum limit to avoid doubling we want to be in the first "circle" with $-2 < M < 0$. For finite lattice spacing this range will be renormalized by gauge fields.

Since D_W is not in general normal, there seem to be two different unitary operators one could consider here, i.e. $V_1 = D_W(D_W^\dagger D_W)^{-1/2}$ and $V_2 = (D_W D_W^\dagger)^{-1/2} D_W$. These are actually equal:

$$\begin{aligned} V_1 &= D_W(D_W^\dagger D_W)^{-1/2} \\ &= \gamma_5 \left(\gamma_5 D_W \left((\gamma_5 D_W)^\dagger \gamma_5 D_W\right)^{-1/2}\right) \\ &= \gamma_5 \left(\left((\gamma_5 D_W)^\dagger \gamma_5 D_W\right)^{-1/2} \gamma_5 D_W\right) \\ &= (D_W D_W^\dagger)^{-1/2} D_W \\ &= V_2 \\ &= \gamma_5 \mathrm{sign}(\gamma_5 D_W) \\ &= \gamma_0 \mathrm{sign}(\gamma_0 D_W) \end{aligned}$$

where I have used the fact that $\gamma_5 D_W$ is Hermetian, thus commuting with its dagger, and the condition $\gamma_5 D_W \gamma_5 = D_W^\dagger$.

Since V is unitary, D is normal and can be diagonalized, just as for the continuum operator. This unitarity also means that D satisfies a GW like relation

$$\begin{aligned} V^\dagger V = 1 &= 1 - D - D^\dagger + DD^\dagger \\ &= 1 - D - \gamma_5 D \gamma_5 + D\gamma_5 D \gamma_5 \end{aligned}$$

or

$$D\gamma_5 + \gamma_5 D - D\gamma_5 D = 0$$

A similar relation is true with γ_0 replacing γ_5

$$D\gamma_0 + \gamma_0 D - D\gamma_0 D = 0$$

The symmetry between γ_0 and γ_5 seems to be important.

To get this into a nicer form, multiply by γ_0 and define $h = \gamma_0 D$ to give

$$h\gamma_5 - \gamma_5 h - h\gamma_5\gamma_0 h = 0,$$

Now use the fact that $\gamma_0\gamma_5$ anticommutes with h to write this in the form

$$[h, \gamma_5(1 - D/2)] = 0.$$

This hints that $\gamma_5(1 - D/2)$ is what we want for the axial charge. Define

$$Q_5 = \psi^\dagger \gamma_5 (1 - D/2)\psi \equiv \psi^\dagger q_5 \psi.$$

This commutes with our Hamiltonian. Verification:

$$\gamma_0 D\gamma_5 - \gamma_0 D\gamma_5 D/2 - \gamma_5\gamma_0 D + \gamma_5 D\gamma_0 D/2 = \gamma_0(D\gamma_5 + \gamma_5 D - D\gamma_5 D) = 0$$

by the HGW relation.

This is an exact symmetry of the Fermion part of the Hamiltonian. However Q_5 involves link variables and thus does not commute with the kinetic part of the gauge field Hamiltonian. This must be crucial for anomalous processes.

The energy and chiral charges of individual fermion states are highly correlated. To see this note that

$$D = \gamma_0 h = 2 - 2\gamma_5 q_5$$

Rearranging

$$\gamma_0 h + 2\gamma_5 q_5 = 22$$

This times its dagger gives

$$h^2/4 + q_5^2 = 1.$$

Thus the remarkable result that the single fermion states lie on a circle in the $E/2$, Q_5 plane! The doublers all appear around $E = 2$, thus for them Q_5 vanishes. Unlike γ_5, the eigenvalues of q_5 are not ± 1.

Applying $\gamma_0\gamma_5$ to an eigenvector flips the sign of both E and q_5, showing that eigenvalues pair across diagonals of this circle. When no gauge fields are present $\phi \to \gamma_5\phi^*$ will flip the sign of the energy without changing the axial charge. Then the states form a quartet lying at $\pm E, \pm q_5$. Then the axial charge of the vacuum, when negative energy states are filled, will vanish. When gauge fields are present, the link variables reduce this four fold symmetry to the diagonal one only and the vacuum can have axial charge.

Now we can see how the anomaly works for the $U(1)$ one D case, paralleling almost exactly what Ivan and I did. The gauge field effectively shifts the allowed momenta of $p = 2\pi n/L$ to $p = 2\pi N/L + \alpha/L$. As alpha runs from 0 to 2π, we pass through the analogue of a tunnelling event. All states of a given n step forward to the next one.

Start with a single particle state of low momentum and, say, positive chirality. Then as the momentum goes from 0 to π, the axial charge decreases from 1 to zero, and then continues to decrease to -1 at 2π. When h and q_5 are simultaneously diagonalized, the periodicity in momentum is doubled; as q runs from 0 to 4π all states are smoothly connected. This motion is just like what Ivan and I did, except now the motion is entirely on a circle.

In the adiabatic anomaly process mentioned to above, the modes passing through zero are related to the zero modes in the Lagrangian formulation. To see this more explicitly, consider adiabatically changing the Hamiltonian so $H = H(t) = \psi^\dagger h(t)\psi$ where t runs from say $-T/2$ to $T/2$ and T is some large number. In particular, assume

$$\frac{d}{dt}h(t) = O(1/T)$$

Then consider the i'th eigenvector $\chi_i(t)$ satisfying

$$h(t)\chi(t) = E_i(t)\chi(t)$$

I assume no malicious time dependent phases are inserted so that

$$\frac{d}{dt}\chi(t) = O(1/T)$$

Now construct

$$\phi(t) = e^{-E(t)t}\chi(t)$$

This satisfies

$$\left(\frac{d}{dt} + h(t)\right)\phi(t) = O(1/T)$$

or in another form

$$\left(\gamma_0\frac{d}{dt} + D(t)\right)\phi(t) = O(1/T).$$

We have an approximate zero mode of the Euclidian Dirac operator if this is normalizable. This is the case if

$$\lim E(t) \begin{array}{ll} > 0, & t \to T/2 \\ < 0, & t \to -T/2 \end{array}$$

If χ is normalized, then the norm of ϕ is

$$\int_{-T/2}^{T/2} dt\ e^{-2tE(t)}$$

Note that there is a related wave function if we reverse the flow of time

$$\tilde{\phi}(t) = e^{+E(t)t}\chi(t)$$

This satisfies

$$\left(-\frac{d}{dt} + h(t)\right)\tilde{\phi}(t) = O(1/T)$$

and is normalizable if

$$\lim E(t) \begin{array}{l} < 0,\ t \to T/2 \\ > 0,\ t \to -T/2 \end{array}$$

Is there some generalization of this that avoids the adiabatic assumption? There ought to be since the continuum index theorem has no such assumption. Note that the four dimensional operator $D_4 \equiv \left(\gamma_0 \frac{d}{dt} + D(t)\right)$ satisfies the usual hermeticity condition $\gamma_5 D_4 \gamma_5 = D_4^\dagger$. This means that eigenvalues should occur in complex conjugate pairs, and any isolated zero modes must remain real in their eigenvalues.

Questions:

1. How does isospin fit in. The above chiral charge does not commute with the gauge kinetic term, but that is expected since we don't want an exact singlet chiral symmetry. Does there exist an exactly conserved flavored chiral charge?

2. How do we hook things up for a chiral theory with anomaly cancellation, say the 21111 model. Everything is totally finite and well defined so far; can this be continued?

To parallel my continuum understanding of this problem, I want to couple the gauge field to a sum of local chiral currents for the various species and then get nice commutation relations when anomalies cancel. But these are non-local currents, so I'm not sure where to go next.